DOMINATED ESTIMATES OF CONVEX COMBINATIONS OF COMMUTING ISOMETRIES*

BY

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ABSTRACT

The principal result of this paper is that the convex combination of two positive, invertible, commuting isometries of $L_p(X, \mathscr{F}, \mu)$ $1 < p < +\infty$, one of which is periodic, admits a dominated estimate with constant $p/p-1$. In establishing this, the following analogue of Linderholm's theorem is obtained: Let σ and ε be two commuting non-singular point transformations of a Lebesgue Space with τ periodic. Then given $\varepsilon > O$, there exists a periodic non-singular point transformation σ' such that σ' commutes with τ and $\mu(x:\sigma'x \neq \sigma x\} < \varepsilon$. By an approximation argument, the principal result is applied to the convex combination of two isometries of L_p (0, 1) induced by point transformations of the form $\tau x = x^k$, $k > 0$ to show that such convex combinations admit a dominated estimate with constant $p/p-1$.

1. Introduction

In what follows we assume p fixed, $1 < p < +\infty$. Let $(X, \mathscr{F}, \mu)$ be a σ-finite measure space, and let T be a linear operator mapping $L_p(X, \mathscr{F}, \mu)$ into $L_p(X, \mathscr{F}, \mu)$. If there exists a constant $c > 0$ such that

$$\int \sup_n \left(|f|^p, \left|\frac{f + Tf}{2}\right|^p, \cdots, \left|\frac{f + \cdots + T^{n-1}f}{n}\right|^p, \cdots \right) d\mu \leqq c^p \int |f|^p d\mu$$

for each $f \in L_p(X, \mathscr{F}, \mu)$, then we say that T admits of a dominated estimate with constant c. If $\| Tf \|_p = \|f\|_p$ for each $f \in L_p(X, \mathscr{F}, \mu)$, then we say that T is an isometry. If T maps non-negative functions to non-negative functions, then we say that T is positive.

Our main result is that a convex combination of two positive, invertible, commuting isometries, one of which is periodic, admits of a dominated estimate with constant $p/p-1$. To establish this, we will prove an analogue of Lin-

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derholm's Theorem to show that if τ_1 and τ_2 are commuting non-singular point transformations with τ_2 periodic (see Section 2 for definitions), then for every $\varepsilon > 0$, there exists a periodic non-singular point transformation τ_ε such that τ_ε commutes with τ_2 and $\mu\{x: \tau_\varepsilon x \neq \tau_1 x\} < \varepsilon$. In Section 3, we apply the principal result to show that a convex combination of isometries of $L_p(0,1)$ of the form $Tf(x) = f(x^k) \cdot (kx^{k-1})^{1/p}$ admits of a dominated estimate with constant $p/p-1$

2. An analogue of Linderholm's theorem

In this section we will assume that $(X, \mathscr{F}, \mu)$ is a Lebesgue space, i.e., that it is separable, complete, non-atomic, and $\mu(X) = 1$. Let τ be a point transformation of X into itself. If τ is one-to-one, measurable in the sense that $\tau A \in \mathscr{F}$ if and only if $A \in \mathscr{F}$, and if $\mu(\tau A) = 0$ if and only if $\mu(A) = 0$, we say that τ is non-singular. If there exists an integer N such that for almost all $x \in X$ we have $\tau^N x = x$, we say that τ is periodic. If there exists an integer n such that for almost all x belonging to a set A we have $\tau^n x = x$, where n is the least such integer, we say that τ has period n on A.

The main result of this section is the following:

THEOREM 2.1. *Let τ and σ be two non-singular point transformations of the Lebesgue space $(X, \mathscr{F}, \mu)$ with τ periodic. Then given $\varepsilon > 0$, there exists a periodic non-singular point transformation σ^1 of $(X, \mathscr{F}, \mu)$ such that σ^1 commutes with τ and*

$$\mu\{x: \sigma^1 x \neq \sigma x\} < \varepsilon.$$

This is a generalization of Linderholm's approximation theorem:

LINDERHOLM'S APPROXIMATION THEOREM. *Let σ be a non-singular point transformation of the Lebesgue space $(X, \mathscr{F}, \mu)$ and let $\varepsilon > 0$. Then there exists a periodic point transformation τ such that*

$$\mu\{x: \tau x \neq \sigma x\} < \varepsilon.$$

In [3], p. 71, there is a proof of this theorem in the measure preserving case that is easily adaptable to the non-singular case.

The bulk of the proof of Theorem 2.1 is contained in the following three lemmas.

LEMMA 2.1. *Let τ and σ be two commuting non-singular point transformations of the Lebesgue space $(X, \mathscr{F}, \mu)$ such that τ is periodic with period n and σ is anti-periodic. Then for every integer m, there exists a measurable set A of positive measure such that the sets*

$$A, \sigma A, \cdots, \sigma^{m-1}A, \tau A, \sigma\tau A, \cdots, \sigma^{m-1}\tau A, \cdots, \tau^{n-1}A, \sigma\tau^{n-1}A, \cdots, \sigma^{m-1}\tau^{n-1}A$$

are all disjoint.

PROOF. We show that if $k < n - 1$, the existence of a set A_{k-1} of positive measure such that the sets

$$A_{k-1}, \sigma A_{k-1}, \cdots, \sigma^{m-1}A_{k-1}, \tau A_{k-1}, \ \sigma\tau A_{k-1}, \cdots, \sigma^{m-1}\tau A_{k-1}, \cdots,$$
$$\tau^{k-1}A_{k-1}, \sigma\tau A_{k-1}, \cdots, \sigma^{m-1}\tau A_{k-1}, \cdots, \tau^{k-1}A_{k-1},$$

are all disjoint implies the existence of a set A_k of positive measure such that $A_k, \cdots, \sigma^{m-1}A_k, \tau A_k, \cdots, \sigma^{m-1}A_k, \cdots, \tau^k A_k, \cdots, \sigma^{m-1}\tau^k A_k$ are all disjoint. This will establish the lemma by induction since Linderholm's approximation theorem yields the existence of A_0.

We proceed in two steps:

(i) We show the existence of A_{k-1} implies that there exists a subset B of A_{k-1} of positive measure such that $\tau^k B, B, \sigma B, \cdots, \sigma^{m-1}B$ are all disjoint. This also implies that $\tau^k B, B, \sigma B, \cdots, \sigma^{m-1}B, \tau B, \cdots, \sigma^{m-1}\tau B, \cdots, \tau^{k-1}B, \cdots, \sigma^{m-1}\tau^{k-1}B$ are all disjoint.

(ii) We show that the existence of a set B such that $B, \sigma B, \cdots, \sigma^{m-1}B, \cdots, \tau^{k-1}B, \cdots, \sigma^{m-1}\tau^{k-1}B, \tau^k B, \cdots, \sigma^{l-1}\tau^k B$, $1 < l < m$, are all disjoint implies the existence of a subset C of B of positive measure such that $C, \cdots, \sigma^{m-1}C, \cdots, \sigma^{m-1}\tau^{k-1}C, \tau^k C$ $\cdots, \sigma^{l-1}\tau^k C, \sigma^l \tau^k C$ are all disjoint. This and (i) implies the existence of A_k.

In both (i) and (ii), we use the fact that if two transformations σ and τ are such that $\sigma(A) = \tau(A)$ (modulo a null set) for every $A \in \mathscr{F}$, then σ and τ are the same almost everywhere. To see this, note that $\sigma(A) = \tau(A)$ (modulo null sets) implies that $\tau^{-1}\sigma$ is equal to the identity as a set transformation (modulo null sets), and hence is a measure preserving transformation. But since $(X, \mathscr{F}, \mu)$ was assumed to be a Lebesgue space, this implies that $\tau^{-1}\sigma$ is isomorphic to the identity point transformation (see [3], or [1, pp. 69–70]).

(i) That τ is periodic and σ antiperiodic implies that there exists a subset B_1 of A_{k-1} such that $\mu(\tau^k B_1 \Delta B_1) \neq 0$. If $\mu(B_1 - \tau^k B_1) \neq 0$, put $C_1 = B_1 - \tau^k B_1$. Otherwise, put $C_1 = \tau^{-k}(\tau^k B_1 - B_1)$. Then C_1 and $\tau^k C_1$ are disjoint. Since τ is nonsingular, we have $\mu(C_1) > 0$.

Continuing by induction, suppose we have a subset C_i of A_{k-1} of positive measure such that $C_i, \sigma C_i, \cdots, \sigma^{i-1}C_i, \ \tau^k C_i, \ i < m$, are all disjoint. Since σ is antiperiodic, there exists a subset B_{i+1} of C_i of positive measure such that $\mu(\sigma^i B_{i+1} \Delta \tau^k B_{i+1}) \neq 0$. If $\mu(\sigma^i B_{i+1} - \tau^k B_{i+1}) \neq 0$, put $C_{i+1} = \sigma^{-i}(\sigma^i B_{i+1} - \tau^k B_{i+1})$. Otherwise, put

$C_{i+1} = \tau^{-k}(\tau^k B_{i+1} - \sigma_i B_{i+1})$. Since τ and σ are non-singular, $\mu(C_{i+1}) > 0$. (i) is established by putting $B = C_m$.

(ii) The only place $\sigma(\sigma^{l-1}\tau^k B)$ can intersect

$$\left(\bigcup_{\substack{0 \leq j \leq m-1 \\ 0 \leq i \leq k-1}} \sigma^j \tau^i B\right) \bigcup \left(\bigcup_{j=0}^{l-1} \sigma^j \tau^k B\right)$$

is on $\bigcup_{i=0}^{k} \tau^i B$. Since σ and τ commute, this intersection must also be restricted to $\bigcup_{i=0}^{m-1} \sigma^i B$. Since τ is periodic and σ is antiperiodic, there must exist a subset G of B of positive measure such that $\mu(G \Delta \tau^k \sigma^l G) \neq 0$. If $\mu(G - \tau^k \sigma^l G) \neq 0$, put $C = G - \tau^k \sigma^l G$. Otherwise put $C = \tau^{-k}\sigma^{-l}(\tau^k \sigma^l G - G)$. Again, that σ and τ are non-singular implies $\mu(C) > 0$.

We remark that the above proof applies to any pair of commuting point transformations such that for every pair of integers $k, l, 0 \leqq k \leqq m, 0 \leqq l \leqq n, kl \neq 0$, we have $\mu\{x : \tau^k x = \sigma^l x\} = 0$, $\mu\{x : \tau^k x = x\} = 0$.

LEMMA 2.2. *In Lemma 2.1, the set A (for a fixed m) may be chosen to be maximal in the sense that if* $A \subset B$, $\mu(B) > \mu(A)$, *then there exist* i, j, k, l, $1 \leqq i \leqq n-1$, $1 \leqq j \leqq n-1$, $1 \leqq k \leqq m-1$, $1 \leqq l \leqq m-1$, *such that* $\mu(\tau^i \sigma^k B \bigcap \tau^j \sigma^l B) > 0$, *where either* $i \neq j$ *or* $k \neq l$, *or both.*

PROOF. Consider the family of collections $\mathscr{A}$ of sets E_α, where $E_\alpha \in \mathscr{F}$, $\mu(E_\alpha) > 0$, and

$$\left(\left(\bigcup_{\substack{0 \leq i \leq n-1 \\ 0 \leq j \leq m-1}} \tau^j \sigma^i E_\alpha\right) \bigcap \left(\bigcup_{\substack{0 \leq i \leq n-1 \\ 0 \leq j \leq m-1}} \tau^j \sigma^i E_\beta\right)\right) = \varnothing$$

for $\alpha \neq \beta$. Partially order such $\mathscr{A}$'s by inclusion. The Hausdorff maximal principle yields a maximal collection $\mathscr{A}_1$. Since $\mu(X) < \infty$, $\mathscr{A}_1$contains at most a countable number of sets; thus $A = \bigcup_{E_\alpha \in \mathscr{A}_1} E_\alpha \in \mathscr{F}$. Finally, suppose $B \supset A$, $\mu(B) > \mu(A)$, and $B, \sigma B, \cdots, \sigma^{m-1}B, \tau B, \sigma\tau B, \cdots, \sigma^{m-1}\tau B, \cdots, \tau^{n-1}B, \sigma\tau^{n-1}B, \cdots, \sigma^{m-1}\tau^{n-1}B$ are all disjoint. The collection $\mathscr{A}_2 = \mathscr{A}_a \bigcup \{B - A\}$ is again of the sort considered and properly majorizes $\mathscr{A}_1$, contradicting the maximality of $\mathscr{A}_1$.

LEMMA 2.3. *Let* τ *and* σ *be two non-singular point transformations of the Lebesgue space* $(X, \mathscr{F}, \mu)$, *where* τ *has period n and* σ *is antiperiodic. Then given an integer m and* $\varepsilon > 0$, *there exists a measurable set F of positive measure such that the sets* $F, \sigma F, \cdots, \sigma^{m-1}F$, $\tau F, \sigma\tau F, \cdots, \sigma^{m-1}\tau F, \cdots, \tau^{n-1}F$, $\sigma\tau^{n-1}F, \cdots, \sigma^{m-1}$ $\tau^{n-1}F$ *are all disjoint and*

$$\mu\left(X - \bigcup_{\substack{0 \leqq i \leqq m-1 \\ 0 \leqq j \leqq n-1}} \tau^j \sigma^i F\right) < \varepsilon.$$

PROOF. Choose an integer p so large that $1/p < \varepsilon$. By Lemma 2.2, choose a maximal set A of positive measure such that the sets $A, \sigma A, \cdots, \sigma^{mp-1}A, \tau A, \sigma\tau A$ $\cdots, \sigma^{mp-1}\tau A, \cdots, \tau^{n-1}A, \sigma\tau^{n-1}A, \cdots, \sigma^{mp-1}\tau^{n-1}A$ are all disjoint.

Since σ and τ commute, if B is a subset of $\sigma^{mp-1}\tau^i A$, $0 \leqq i \leqq n-1$ such that $\sigma^k B$ is disjoint from

$$C = \bigcup_{\substack{0 \leqq i \leqq mp-1 \\ 0 \leqq j \leqq n-1}} \sigma^i \tau^j A,$$

then for every j, $0 \leqq j \leqq n-1$, $\sigma^k \tau^j B$ is disjoint from C. Therefore, there can exist no subset D of $\bigcup_{j=0}^{n-1} \sigma^{mp-1}\tau^j A$ of positive measure such that $\sigma^k D$ is disjoint from C for all k, $1 \leqq k \leqq l$ where $l > mp-1$, for otherwise $\sigma(D \bigcap \sigma^{mp-1}A)$ would provide a way of enlarging A, which contradicts the maximality of A.

Let E_{ik} be that subset of $\tau^i A$, $0 \leqq i \leqq n-1$, such that k is the least positive integer such that $\sigma^{k+mp-1}E_{ik}$ intersects C (and hence $\bigcup_{i=0}^{n-1}\tau^i A$). For each i, $1 \leqq i \leqq n-1$, $\mu(\tau^i A - \bigcup_{k=1}^{mp} E_{ik}) = 0$, for otherwise $\sigma^{-mp}(A - \bigcup_{k=1}^{mp} E_{0k}) = D_1$ would be a set of positive measure such that D_1, $\sigma D_1, \cdots, \sigma^{mp-1}D_1, \tau D_1, \sigma\tau D_1,$ $\cdots, \sigma^{mp-1}\tau D_1, \cdots, \tau^{n-1}D_1, \sigma\tau^{n-1}D_1, \cdots, \sigma^{mp-1}\tau^{n-1}D_1$, would be disjoint from each other and from C, so that D_1 would provide a way of enlarging A, again contradicting the maximality of A.

Thus $\mu(X - C) = 0$, since we now have that C is invariant under τ and σ; for otherwise we could find a subset A_1 of positive measure of $X - C$ with the same property as A which would contradict the maximality of A.

Put

$$S_k = \bigcup_{i=0}^{n-1} E_{ik}, \quad G_k = \bigcup_{i=0}^{mp+k-2} \sigma^i S_k,$$

$$H_{ik} = \bigcup_{j=(i-1)m}^{im-1} \sigma^j S_k \quad i < \left[\frac{mp+k-2}{m}\right]$$

$$= \bigcup_{j=(i-1)}^{im-1} \sigma^j S_k \quad i = \left[\frac{mp+k-2}{m}\right]$$

where $[x]$ is the largest integer $\leqq x$.

Now for every k, at least one of the $H_{i,k}$ has measure less than $p^{-1}\mu(G_k)$, since there are at least p $H_{i,k}$. Choose one of these, $H_{l\,k}$ say. Put

$$F_k = \left(\bigcup_{i=0}^{l-2} \sigma^{im} E_{0k}\right) \bigcup \left(\bigcup_{i=0}^{p+[(k+1)/m]-l} \sigma^{(i+l-1)m+t} E_{0k}\right)$$

where $t = k - 1 - m[k - 1/m]$. Note that F_k is constructed by taking E_{0k} and each "mth" iterate under σ of E_{0k} until one iterate is contained in $H_{l-1,k}$. We then add to F_k all the "mth" iterates under σ^{-1} of $\sigma^{mp-1+k-1}E_{0k}$ until one of these iterates is contained in $H_{l,k}$. Thus the sets $F_k, \sigma F_k, \cdots, \sigma^{m-1}F_k, \tau F_k, \sigma\tau F_k, \cdots, \sigma^{m-1}\tau F_k, \cdots, \tau^{n-1}F_k, \sigma\tau^{n-1}F_k, \cdots, \sigma^{m-1}\tau^{n-1}F_k$ are all disjoint, and if

$$\bigcup_k = \bigcup_{\substack{0 \leqq i \leqq m-1 \\ 0 \leqq j \leqq n-1}} \tau^j \sigma^i F_k,$$

$(G_k - \bigcup_k) \subset H_{l,k}$.

If $F = \bigcup_{k=1}^{mp} F_k$, then $F, \sigma F, \cdots, \sigma^{m-1}F, \tau F, \sigma\tau F, \cdots, \sigma^{m-1}\tau F, \cdots, \tau^{n-1}F, \sigma\tau^{n-1}F, \cdots, \sigma^{m-1}\tau^{n-1}F$ are all disjoint, and

$$\mu\left(X - \bigcup_{\substack{0 \leqq i \leqq m-1 \\ 0 \leqq j \leqq n-1}} \sigma^i \tau^j F\right) \leqq \sum_{k=1}^{mp} \frac{1}{p} \mu(G_k) < \varepsilon. \qquad \text{Q.E.D.}$$

PROOF OF THEOREM 2.1. We first note that since τ and σ commute, the sets on which either τ or σ has period k, for any integer k, are each invariant under both τ and σ. Let n be the least integer such that $\tau^n x = x$ for almost all $x \in X$. If we normalize those sets of positive measure A_k where τ has period $k \leqq n$ and show there exists σ_k^1 defined on A_k such that σ_k^1 commutes with τ and $\mu_k\{x: \sigma_k^1 x \neq \sigma x\} < \varepsilon$, where μ_k is μ normalized so that A_k has measure one, then we may define σ^1 to be σ_k^1 on A_k, and have that σ^1 commutes with τ and

$$\mu\{x: \sigma^1 x \neq \sigma x\} = \sum_k \mu\{x: x \in A_k^1, \sigma_k x \neq \sigma x\} < \sum_k \varepsilon\mu(A_k) < \varepsilon$$

where the sums extend over all k such that A_k is defined. Thus we may assume without loss of generality that τ has period n.

Let B_k be the set on which σ has period k. Since $\mu(X) = 1$, there must exist integer N such that $\mu(_{k=N}^{\infty} B_k) < \varepsilon/2$. Define σ^1 to be σ on B_k, $1 \leqq k \leqq N$, and the identity on B_k, $k > N$.

Let m be an integer such that $1/m < \varepsilon/4$, and choose $\delta > 0$ such that for every measurable set A with $\mu(A) < \delta$, we have $\mu(A \bigcup \sigma^{-1}A \bigcup \cdots \bigcup \sigma^{-m}A) < \varepsilon/4$. $W = (X - \bigcup_{k=1}^{\infty} B_k)$ is invariant under both τ and σ. If $\mu(W) > 0$, we may apply Lemma 2.3 to find a subset A of W such that the sets $A, \sigma A, \cdots, \sigma^{m-1}A, \tau A, \sigma\tau A \cdots, \sigma^{m-1}\tau A, \cdots, \tau^{n-1}A, \sigma\tau^{n-1}A, \cdots, \sigma^{m-1}\tau^{n\ 1}A$ are all disjoint and

$$D = W - \bigcup_{\substack{0 \leqq i \leqq n-1 \\ 0 \leqq j \leqq n-1}} \tau^i \sigma^j A$$

has measure less than δ. At least one of the sets $C_k = \bigcup_{i=0}^{n-1} \sigma^k \tau^i A$ has measure less than $1/m$, C_l say. Put $B = \sigma^{-(m-l-1)} A$. Then the sets $B, \sigma B, \cdots, \sigma^{m-1} B, \tau B, \sigma\tau B, \cdots, \sigma^{m-1}\tau B, \cdots, \tau^{n-1} B, \sigma\tau^{n-1} B, \cdots, \sigma^{m-1}\tau^{n-1} B$ are all disjoint, $\bigcup_{i=0}^{n-1} \sigma^{m-1}\tau^i B = C_l$, and

$$E = W - \bigcup_{\substack{0 \leqq i \leqq n-1 \\ 0 \leqq j \leqq m-1}} \tau^i \sigma^j B \subset \bigcup_{i=0}^{m} \sigma^{-i} D.$$

We define σ^1 on W by

$$\begin{aligned} \sigma^1 x &= \sigma^{-m} x \qquad && x \in C_l \\ &= x && x \in E \\ &= \sigma x && \text{otherwise.} \end{aligned}$$

Then

$$\begin{aligned} \mu\{x : \sigma^1 x \neq \sigma x\} &= \mu(C_l) + \mu(E) + \mu\left(\sum_{k=N}^{\infty} B_k\right) \\ &< \frac{1}{m} + \mu\left\{\bigcup_{i=0}^{m} \sigma^{-i} D\right\} + \frac{\varepsilon}{2} < \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Also, σ^1 commutes with τ, since σ^1 is the identity on E which is invariant under τ, and for $x \in C_l$, $\sigma^{-m} x \in \bigcup_{i=0}^{n-1} \tau^i B$, so $\sigma^{-m}\tau x = \tau\sigma^{-m} x$.

3. Principal result

Recall that we have assumed p fixed, $1 < p < +\infty$. If T is a linear operator of $L_p(X, \mathscr{F}, \mu)$ with norm less than or equal to one, we say that T is a contraction. If there exists $h \in L_p(X, \mathscr{F}, \mu)$, $h > 0$, such that $Th = h$, we say that T has a positive fixed point. In [2], it was shown that positive contractions of $L_0(X, \mathscr{F}, \mu)$ having a positive fixed point admit of a dominated estimate with constant $p/p-1$. This result implies the following lemma.

LEMMA 3.1. *If T_1 and T_2 are commuting periodic positive invertible isometries of $L_p(X, \mathscr{F}, \mu)$, then $T = \alpha T_1 + (1-\alpha) T_2$, $0 < \alpha < 1$, admits of a dominated estimate with constant $p/p-1$.*

PROOF. Since $0 < \alpha < 1$, we have that T is a positive contraction of $L_p(X, \mathscr{F}, \mu)$ That T_1 and T_2 are periodic implies that T_1 and T_2 have positive fixed points, h_1

and h_2 say. Let n be the period of T_1, m the period of T_2. Since for every integer k we have $T_2^k h_1$ is a fixed point of T_1 and $T_1^k h_2$ is a fixed point of T_2, we have that

$$h = h_1 + T_2 h_1 + \cdots + T_2^{m-1} h_1 + h_2 + T_1 h_2 + \cdots + T_1^{n-1} h_2$$

is a positive fixed point of both T_1 and T_2. Hence h is also a positive fixed point of T_1 and T admits of a dominated estimate with constant $p/p-1$.

It is a result essentially due to Banach that the positive invertible isometries of $L_p(X, \mathscr{F}, \mu)$ can be represented in the form

$$Tf = (f \circ \tau) \left(\frac{d\mu \circ \tau}{d\mu} \right)^{1/p},$$

where $f \circ \tau$ means $(f \circ \tau)(x) = f(\tau x)$, $\mu \circ \tau$ is the measure defined by $(\mu \circ \tau)(A) = \mu(\tau A)$, and where τ is a non-singular point transformation of $L_p(X, \mathscr{F}, \mu)$ (see [4]). In order to apply the results of the previous section, we prove the following two lemmas to show that if T is periodic then so is its associated point transformation, and that if two positive invertible isometries commute, so do their associated point transformations.

Lemma 3.2. *Let T be a periodic positive invertible isometry of $L_p(X, \mathscr{F}, \mu)$, where $(X, \mathscr{F}, \mu)$ is a Lebesgue space. Then T is of the form*

$$Tf = (f \circ \tau) \left(\frac{d\mu \circ \tau}{d\mu} \right)^{1/p}$$

where τ is a periodic non-singular point transformation of $(X, \mathscr{F}, \mu)$.

Proof. For every $A \in \mathscr{F}$ and positive integer k, we have

$$\int_A \left(\frac{d\mu \circ \tau^k}{d\mu} \circ \tau \right) \frac{d\mu \circ \tau}{d\mu} d\mu = \int_A \left(\frac{d\mu \circ \tau^k}{d\mu} \right) \circ \tau d(\mu \circ \tau)$$

$$= \int_{\tau A} \frac{d\mu \circ \tau^k}{d\mu} d\mu = \mu \circ \tau^k(\tau A).$$

Therefore, we have that

$$\left(\frac{d\mu \circ \tau^k}{d\mu} \circ \tau \right) \frac{d\mu \circ \tau}{d\mu} = \frac{d\mu \circ \tau^{(k+1)}}{d\mu},$$

and by induction,

$$T^k f = (f \circ \tau^k) \left(\frac{d\mu \circ \tau^k}{d\mu} \right)^{1/p}.$$

Let T have period n. Since $\mu(X) < \infty, 1 \in L_p(X, \mathscr{F}, \mu)$ and $T^n(1) = (d\mu \circ \tau^n / d\mu)^{1/p}$ so for every $A \in \mathscr{F}$, $T^n(\chi_A) = (\chi_A \circ \tau^n) = \chi_A$, so that $(X, \mathscr{F}, \mu)$ a Lebesgue space implies $\tau^n x = x$ for almost every x.

LEMMA 3.3. *Let T_1 and T_2 be two commuting positive invertible isometries of $L_p(X, \mathscr{F}, \mu)$ of the form,*

$$T_1 f = (f \circ \tau_1)\left(\frac{d\mu \circ \tau_1}{d\mu}\right)^{1/p}$$

$$T_2 f = (f \circ \tau_2)\left(\frac{d\mu \circ \tau_2}{d\mu}\right)^{1/p}$$

where τ_1 and τ_2 are non-singular point transformations of the Lebesgue space $(X, \mathscr{F}, \mu)$. Then τ_1 and τ_2 commute.

PROOF. First note that for every $A \in \mathscr{F}$,

$$\int_A \frac{d\mu \circ \tau_2}{d\mu} \circ \tau_1 d\mu \circ \tau_1 = \int_{\tau_1 A} \frac{d\mu \circ \tau_2}{d\mu} d\mu = \mu(\tau_2 \tau_1 A)$$

that so

$$\frac{d\mu \circ \tau_2}{d\mu} \circ \tau_1 = \frac{d\mu \circ \tau_2 \tau_1}{d\mu \circ \tau_1}.$$

and

$$T_1 T_2 f = (f \circ \tau_2 \circ \tau_1)\left(\frac{d\mu \circ \tau_2}{d\mu} \circ \tau_2 \cdot \frac{d\mu \circ \tau_1}{d\mu}\right)^{1/p} = (f \circ \tau_2 \tau_1)\left(\frac{d\mu \circ \tau_2 \tau_1}{d\mu}\right)^{1/p},$$

and similarly

$$T_2 T_1 f = f \circ \tau_1 \tau_2 \left(\frac{d\mu \circ \tau_1 \tau_2}{d\mu}\right)^{1/p}.$$

Now $\mu(X) < \infty$, so $1 \in L_p(X, \mathscr{F}, \mu)$ and since T_1 and T_2 commute, we have

$$\frac{d\mu \circ \tau_2 \tau_1}{d\mu} = \frac{d\mu \circ \tau_1 \tau_2}{d\mu}.$$

Therefore, if $f = \chi_A$, $A \in \mathscr{F}$, we have

$$\chi_{\tau_1 \tau_2 A} = \chi_{\tau_2 \tau_1 A},$$

for almost all $x \in X$, so that $(X, \mathscr{F}, \mu)$ a Lebesgue space implies $\tau_1 \tau_2 x = \tau_2 \tau_1 x$ for almost all $x \in A$.

We now prove our main result. Note that the restriction that $\mu(X) < \infty$ that is inherent in the assumption that $(X,\mathscr{F},\mu)$ is a Lebesgue space can be lifted by considering operators of the form $T_nf=\chi_{A_n}\cdot(T(\chi_{A_n}f)),\ \mu(A_n)<\infty,\ \lim_{n\to\infty}A_n=X$.

THEOREM 3.1. *Let $(X,\mathscr{F},\mu)$ be a Lebesgue space and let T be a contraction of $L_p(X,\mathscr{F},\mu)$ of the form*

$$Tf=\alpha T_1f+(1-\alpha)T_2f,\qquad 0<\alpha<1,$$

where T_1 and T_2 are commuting positive invertible isometries of $L_p(X,\mathscr{F},\mu)$ and T_1 is periodic. Then T admits of a dominated estimate with constant $p/p-1$.

PROOF. We may write

$$T_1f=(f\circ\tau_1)\left(\frac{d\mu\circ\tau_1}{d\mu}\right)^{1/p}$$

$$T_2f=(f\circ\tau_2)\left(\frac{d\mu\circ\tau_2}{d\mu}\right)^{1/p}$$

where τ_1 and τ_2 are non-singular point transformations of $(X,\mathscr{F},\mu)$. By Lemma 3.2 τ_1 is periodic, and by Lemma 3.3 τ_1 and τ_2 commute. Hence, we may apply Theorem 2.1 and for every $\varepsilon>0$, find a non-singular point transformation τ_ε such that

$$\mu\{x:\tau_\varepsilon x\neq\tau_2x\}<\varepsilon$$

and τ_ε is periodic and commutes with τ_2. By Lemma 4.2 of [2], the positive invertible isometry of $L_p(X,\mathscr{F},\mu)$ defined by

$$T_\varepsilon f=(f\circ\tau_\varepsilon)\left(\frac{d\mu\circ\tau_\varepsilon}{d\mu}\right)^{1/p}$$

is periodic, so that by Lemma 3.1, the contraction u_ε of $L_p(X,\mathscr{F},\mu)$ defined by

$$\mu_\varepsilon=\alpha T_1f+(1-\alpha)T_\varepsilon f$$

admits of a dominated estimate with constant $p/p-1$.

Now the operators T_ε approximate T_2 in the strong operator topology (see [4]), so the operators u_ε approximate T in the strong operator topology. The theorem now follows since if $\{T_n\}$ and T are operators of $L_p(X,\mathscr{F},\mu)$ such that $\{T_n\}$ converges to T in the strong operator topology, and such that each T_n admits of a dominated estimate with constant c, then T admits of a dominated estimate with constant c as well.

4. An application

In this section we show that the convex combination of two positive invertible isometries of $L_p(0, 1)$, which are induced by point transformations τ_k defined by $\tau_k x = x^k$, admit of a dominated estimate with constant $p/p - 1$. Here we cannot use Theorem 3.1 directly, because such isometries are not periodic. However Lemma 4.1 will show that we can approximate the two isometries simultaneously with positive invertible isometries that commute and such that one of them is periodic.

LEMMA 4.1. *Let τ and σ be two non-singular point transformations of* $(0,1)$. *If τ and σ are of the form*

$$\sigma x = x^k$$

$$\tau x = x^l, \qquad k \neq l$$

where k and l are positive real numbers, then given $\varepsilon > 0$, there exists non-singular point transformations τ_1 and σ_1 such that τ_1 and σ_1 commute,

$$\mu\{x : \sigma_1 x \neq \sigma x\} < \varepsilon$$

$$\mu\{x : \tau_1 x \neq \tau x\} < \varepsilon,$$

and one of τ_1, σ_1 is periodic.

PROOF. We will show that if $k > 1$, $l > 1$, then there exist commuting transformations σ_1, τ_1, such that

$$\mu\{x : \sigma_1 x \neq \sigma x\} < \varepsilon$$

$$\mu\{x : \sigma_1^{-1} x \neq \sigma^{-1} x\} < \varepsilon$$

$$\mu\{x : \tau_1 x \neq \tau x\} < \varepsilon$$

$$\mu\{x : \tau_1^{-1} x \neq \tau^{-1} x\} < \varepsilon.$$

This will establish the assertion, since if k, say, is less than 1, we may approximate σ^{-1} and τ in this fashion by σ_1 and τ and have σ_1^{-1} commuting with $\tau, \sigma_1{}^{-1}$ periodic if σ_1 is, and

$$\mu\{x : \sigma_1^{-1} x \neq \sigma x\} < \varepsilon.$$

Since $k \neq l$, we will assume with no less of generality that $l < k$. Let c be a positive real number less than 1 and such that $1 - c^k < \varepsilon/2$. Choose an integer n such that $c^{k^{n-1}} < \varepsilon/2$. Denote by A the interval $(c^k, 1)$, by B the interval $(0, c^{k^n})$, and by D_i the intervals $(c,^{k^i} c^{k^{i-1}})$. Note that for every $i < n$, $\sigma D_i = D_{i+1}$. Then for every i, we have $\tau(D_i) \subset D_i \bigcup D_{i+1}$ for every $i < n$ since $\tau(D_i) = (c^{lk^i}, c^{lk^{i-1}}) \subset (c^{k^{i+1}}, c^{k^{i-1}}) = D_i \bigcup D_{i+1}$. For every m, $1 \leqq m \leqq n$, put $E_m = D_m - \tau^{-1}D_m$ and $F_m = D_m - \tau D_m$. Since τ and σ commute, we have $\sigma E_m = E_{m+1}$, $\sigma F_m = F_{m+1}$ for every $m < n$. Define τ_1 by

$$\begin{aligned} \tau_1 x &= \sigma^{-n}\tau\sigma^{-1}x && x \in E_n \\ &= x && x \in A \bigcup B \\ &= \tau x && \text{otherwise} \end{aligned}$$

and σ_1 by

$$\begin{aligned} \sigma_1 x &= \sigma^{-n}x && x \in D_n \\ &= x && x \in A \bigcup B \\ &= \sigma x && \text{otherwise.} \end{aligned}$$

Then σ_1 is periodic, and the following remarks show that σ_1 and τ_1 are well defined and commute, establishing the assertion.

1) $\tau E_{n-1} = F_n$, $\tau^{-n}F_n = F_1$, so τ_1 is well defined.

2) If $x \in A \bigcup B \bigcup (\bigcup_{k=1}^{n-2} D_k)$, there is nothing to prove.

3) If $x \in (D_{n-1} - E_{n-1})$, $\sigma x \in (D_n - E_n)$, again there is nothing to prove.

4) If $x \in (D_n - E_n)$, $\sigma_1 \tau x = \sigma^{-k}\tau x$, $\tau\sigma_1 x = \tau\sigma^{-n}x = \sigma^{-n}x$.

5) If $x \in E_{n-1}$, $\sigma x \in E_n$, $\tau x \in D_n$, and

$$\tau_1\sigma_1 x = \tau_1\sigma x = \sigma^{-n}\tau\sigma^{-1}x = \sigma^{-n}\tau x,$$

$$\sigma_1\tau_1 x = \sigma_1\tau x = \sigma^{-n}\tau x.$$

6) If $x \in E_n$, $\tau_1 x \in D_1$ and $\sigma_1\tau_1 x = \sigma(\sigma^{-n}\tau\sigma^{-1}x) = \sigma^{-n}\tau x$, $\tau_1\sigma_1 x = \tau\sigma^{-n}x$.

THEOREM 4.1. *Let T be a contraction of $L_p(0,1)$ defined by*

$$Tf = \alpha T_1 f + (1-\alpha)T_2 f,\ 0 < \alpha < 1,$$

where T_1 and T_2 can be represented by

$$T_i f(x) = f(x^{k_i})\,(k_i x^{k_i - 1})^{1/p}, \qquad i = 1,2.$$

Then T admits of a dominated estimate with constant $p/p-1$.

PROOF. If $k_1 = k_2$, the theorem follows from [4]. Otherwise, by Lemma 4.1 we may approximate T_1 and T_2 in the strong operator topology by positive invertible isometries $T_{1,\varepsilon}$ and $T_{2,\varepsilon}$ such that $T_{1,\varepsilon}$ and $T_{2,\varepsilon}$ commute and one of them is periodic. By theorem 3.1, the contractions $T_\varepsilon = \alpha T_{1,\varepsilon} + (1-\alpha)T_{2,\varepsilon}$

admit of a dominated estimate with constant $p/p-1$. But the contractions T_ε approximate T in the strong operator topology, so T admits of a dominated estimate with constant $p/p-1$.

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